

Loop Quantum Gravity on Non-Compact Spaces

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Abstract

We present a general procedure for constructing new Hilbert spaces for loop quantum gravity on non-compact spatial manifolds. Given any fixed background state representing a non-compact spatial geometry, we use the Gel'fand-Naimark-Segal construction to obtain a representation of the algebra of observables. The resulting Hilbert space can be interpreted as describing fluctuation of compact support around this background state. We also give an example of a state which approximates classical flat space and can be used as a background state for our construction.

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1 Introduction

Remarkable progress has been made in the field of non-perturbative (loop) quantum gravity in the last decade or so and it is now a rigorously defined kinematical theory. One of the most important results in this area is that geometric operators such as area and volume have discrete spectra. However, before loop quantum gravity can be considered a complete theory of quantum gravity, we must show that the discrete picture of geometry that it provides us reduces to the familiar smooth classical geometry in some appropriate limit. One aspect of this is the recovery of the weak-field limit of quantum gravity which is described by gravitons and their interactions.

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In standard perturbative quantum field theory, gravitons are fields which describe the fluctuations of the metric field around some classical “vacuum” metric (usually Minkowski space). The graviton is then a spin-two particle as defined by the representations of the Poincaré group at infinity. Thus, in order to study graviton physics in the context of loop quantum gravity, a basic requirement is the construction of a state corresponding to the Minkowski metric, which in turn necessitates a proper quantum treatment of asymptotically flat spaces. Given this framework one could then construct asymptotic states (corresponding to gravitons) describing fluctuations of the background metric and the action of the generators of the Poincaré group at infinity.

These are non-trivial requirements in loop quantum gravity, which can be thought of as describing excitations of the three-geometry itself. Hence, the ‘zero excitation state’ of the theory corresponds to the metric “ $g_{ab} = 0$ ” and not the Minkowski metric. In this description, Minkowski space-time is a highly excited state of the quantum geometry containing an infinite number of elementary excitations. The situation, in fact, is analogous to that in finite-temperature field theory where the thermal ground state is a highly excited state of the zero-temperature theory which does not even lie in the standard Fock space. Excitations are then constructed by building a representation of the standard algebra of creation and annihilation operators on the thermal vacuum. In this paper, we shall give an analogous construction in loop quantum gravity which enables us to describe fluctuations of essentially compact support around a flat background metric. It is important to note that these fluctuations can be arbitrarily large and hence we are not quantising linearised general relativity. Indeed, our framework aims to identify quantum linearised general relativity as a sector of the non-perturbative theory.

This work is comprised of two main parts. After a brief review of concepts from loop quantum gravity, we describe how new Hilbert spaces for loop quantum gravity applicable to non-compact spaces can be constructed by finding new representations of the standard algebra of observables given some notion of ‘vacuum’ or ‘background’ state.

We proceed to present a detailed example of a background state \mathcal{Q} , which approximates the Euclidean metric on three-space. This state is related to the weave construction, but differs from it as it is peaked not only in the spin-network basis but also in the connection basis. We show that even though this state is not an element of the standard Hilbert space, as is generically the case for states approximating geometries on non-compact spaces, it can be used as a “vacuum” in the above construction to give genuine Hilbert spaces describing fluctuations around this state.

2 The Structure of Loop Quantum Gravity

Canonical general relativity can be written as a theory of a real $SU(2)$ connection over an oriented three-manifold Σ [2,11]. The classical configuration space \mathcal{A} is given by all smooth connections A on a principle $SU(2)$ -bundle³ P over Σ . Since P is trivial, we can use a global cross section to pull back connections to $\mathfrak{su}(2)$ -valued one-forms⁴ A_a^i on Σ . The conjugate variable to the connection is a densitized triad \tilde{E}_i^b which takes values in the lie algebra $\mathfrak{su}(2)$. The triads can be considered as duals to two-forms $e_{abi} \equiv \eta_{abc}\tilde{E}_i^c$. The dynamics of general relativity on spatially compact manifolds is then completely described by the Gauss constraints which generate $SU(2)$ -gauge transformations, the diffeomorphism constraints which generate spatial diffeomorphisms on Σ , and the Hamiltonian constraint, which is the generator of coordinate time evolution. In the non-compact case, true dynamics is generated by the boundary terms of the Hamiltonian.

For compact spatial manifolds Σ , a well defined quantisation procedure for the above setup has been developed, which we review before discussing our extension to the non-compact case. The strategy is to specify an algebra of classical variables $\mathfrak{B}_{\text{aux}}$ and then to seek a representation of this algebra on some auxiliary Hilbert space \mathcal{H}_{aux} . The second step is to obtain operator versions of the classical constraints and to then impose these on the Hilbert space to obtain a reduced space of physical states along with a representation of the subalgebra of observables that commute with the constraints.

2.1 The classical algebra of observables

To obtain the classical algebra of elementary functions which can be implemented in the quantum theory, we need to integrate the canonically conjugate variables, A_a^i and e_{iab} , against suitable smearing fields. In usual quantum field theory, these fields are three-dimensional. However, in canonical quantum general relativity, due to the absence of a background metric, it is more convenient to smear n -forms against n -dimensional surfaces instead of the usual three-dimensional ones [19,6,3].

Configuration observables can be constructed through holonomies of connections. Given an embedded graph Γ which is a collection of n paths $\{\gamma_1, \dots, \gamma_n\} \in \Sigma$, and a smooth function f from $SU(2)^n$ to \mathbb{C} , we can construct cylindrical

³ This bundle arises as the double cover of the principle $SO(3)$ -bundle of frames on Σ .

⁴ Indices i,j,k,\dots denote an internal $\mathfrak{su}(2)$ indices, while a,b,c,\dots are tensor indices.

functions of the connection:

$$\psi_{f,\Gamma}(A) = f(H(A, \gamma_1), \dots, H(A, \gamma_n)).$$

$H(A, \gamma_i) \in \text{SU}(2)$ is the holonomy assigned to the edge γ_i of Γ by the connection $A \in \mathcal{A}$. We denote by \mathfrak{C} , the algebra generated by all the functions of this form. This is the space of configuration variables. To obtain momentum variables, we smear the two-forms e_{abi} against distributional test fields t^i which take values in the dual of $\mathfrak{su}(2)$ and have two dimensional support. This gives us

$$E_{t,S} \equiv \int_S e_{abi} t^i dS^{ab},$$

where S is a two-dimensional surface embedded in Σ . More precisely (c.f. [3]), we require that $S = \bar{S} - \partial\bar{S}$, where \bar{S} is any *compact*, analytic, two dimensional submanifold of Σ .

The elements of \mathfrak{C} and the functions $E_{t,S}$ are the variables that we wish to promote to quantum operators. They form a large enough subset of all classical observables in the sense that they suffice to distinguish phase space points. The algebra of elementary observables, $\mathfrak{B}_{\text{aux}}$, is the algebra generated by the cylindrical functions and the momentum variables, with the choice of Poisson brackets as given in [3].

2.2 The standard representation

The next step in the quantisation procedure is to construct a Hilbert space \mathcal{H}_{aux} on which the algebra of elementary variables $\mathfrak{B}_{\text{aux}}$ is represented. In this subsection, we describe the construction of this Hilbert space [5,9] concentrating on the GNS (Gel'fand-Naimark-Segal) construction since we shall later make crucial use of this technique.

The GNS construction (see, e.g., [14,18,17] for more detailed expositions) allows us to construct a representation of any $*$ -algebra \mathfrak{A} for any given positive linear form (also called a state) ω on this algebra. This is done in three steps:

- (1) Using ω , define a scalar product on \mathfrak{A} , regarded as a linear space over \mathbb{C} , by

$$\langle \mathbf{a} | \mathbf{b} \rangle = \omega(\mathbf{a}^* \mathbf{b}),$$

for $\mathbf{a}, \mathbf{b} \in \mathfrak{A}$. The positivity of ω implies $\langle \mathbf{a} | \mathbf{a} \rangle \geq 0$.

- (2) To obtain a positive definite scalar product, we construct the quotient $\mathfrak{A}/\mathfrak{I}$ of \mathfrak{A} by the null space $\mathfrak{I} = \{\mathbf{a} \in \mathfrak{A} | \omega(\mathbf{a}^* \mathbf{a}) = 0\}$. We denote the equivalence classes in $\mathfrak{A}/\mathfrak{I}$ by $[\mathbf{a}]$ and we have:

$$\langle [\mathbf{a}] | [\mathbf{a}] \rangle \equiv \|\mathbf{a}\|^2 > 0$$

The completion of $\mathfrak{A}/\mathfrak{I}$ in the above norm is the carrier Hilbert space \mathcal{H}_ω for our representation.

- (3) Finally it can be shown that a representation π_ω of \mathfrak{A} on $\mathfrak{A}/\mathfrak{I}$ (which, if \mathfrak{A} is a Banach $*$ -algebra, can be extended continuously to \mathcal{H}_ω) is given by:

$$\pi_\omega(\mathbf{a})\Psi = [\mathbf{a}\mathbf{b}],$$

for $\Psi = \mathbf{b} \in \mathcal{H}_\omega$ and $\mathbf{a} \in \mathfrak{A}$.

Let us now return to our particular problem. We start by constructing a multiplicative representation of \mathfrak{C} on a Hilbert space \mathcal{H}_{aux} , on which momentum operators will be shown to act as derivations. We use the GNS construction to construct the representation of \mathfrak{C} . To obtain a greater degree of control one first introduces a sup norm to complete \mathfrak{C} to a C^* -algebra $\overline{\mathfrak{C}}$:

$$\|\psi_{f,\Gamma}\|_\infty = \sup_{A \in \mathcal{A}} |\psi(A)_{f,\Gamma}|.$$

The key to representing \mathfrak{C} is to define a positive linear form on it. This can be done using the Haar measure dg on $\text{SU}(2)$ as follows[7]

$$\omega(\psi_{f,\Gamma}) = \int d\mu(A) \psi_{f,\Gamma} \equiv \int_{\text{SU}(2)^n} dg_1 \cdots dg_n f(g_1, \dots, g_n), \quad (1)$$

where $g_i \in \text{SU}(2)$. Note that the right hand side does not depend on Γ . Nevertheless, our definition makes sense since, if⁵ $\psi_{f,\Gamma} = \psi'_{f',\Gamma'}$, then $\omega(\psi) = \omega(\psi')$. This allows us to define the (standard) inner product:

$$\langle \psi_1 | \psi_2 \rangle_s = \omega(\psi_1^* \psi_2) = \int_{\text{SU}(2)^n} f_1^*(g_1, \dots, g_n) f_2(g_1, \dots, g_n) dg_1 \cdots dg_n. \quad (2)$$

Here we make use of the fact that if the functions f_1 and f_2 have a different number of arguments, say $f_1 : \text{SU}(2)^m \rightarrow \mathbb{C}$ with $m < n$, we can trivially extend f_1 to a function on $\text{SU}(2)^n$, which does not depend on the last $n - m$ arguments. Since this product is already positive definite, we can proceed

⁵ This holds if Γ consists of analytic paths but extensions to the non-analytic case are possible, c.f. [10].

directly with the completion of $\overline{\mathfrak{C}}$ to obtain our auxiliary Hilbert space \mathcal{H}_{aux} carrying a multiplicative representation of the algebra \mathfrak{C} . \mathcal{H}_{aux} can also be regarded as space of square integrable functions defined with respect to a genuine measure on some completion $\bar{\mathcal{A}}$ of \mathcal{A} as is done in [5].

We are left with the task of representing the momentum variables on this Hilbert space. This done by constructing essentially self adjoint operators $\hat{E}_{t,S}$ on \mathfrak{C} which can be extended to \mathcal{H}_{aux} . These operators are derivations on \mathfrak{C} i.e. linear maps satisfying the Leibnitz rule, which act on functions $\psi_{f,\Gamma} \in \mathfrak{C}$ only at points where Γ intersects the oriented surface S . The precise definition of these operators is not needed for our purposes, but it can be found, e.g., in [3]. This choice of operators gives the correct representation of the classical algebra $\mathfrak{B}_{\text{aux}}$, which provides us with a kinematical framework for canonical quantum gravity. In the following this representation will be referred to as the standard representation π_s . To obtain physical states we need to introduce the quantum constraints and study their action.

2.3 The constraints

The simple geometrical interpretation of the Gauss and diffeomorphism constraints allows us to bypass the attempt to construct the corresponding constraint operators. Instead, we can construct unitary actions of the gauge group \mathcal{G} and the diffeomorphism group \mathcal{D} on \mathcal{H}_{aux} and demand that physical states be invariant under these actions. The imposition of the Hamiltonian constraint is still an open issue and we will not discuss it. In this sense, our entire discussion is at the kinematical level.

The group \mathcal{G} has a natural action on the space of connections which induces a unitary action of \mathcal{G} on \mathcal{H}_{aux} . Gauge invariance is simply achieved by restricting to the subspace $\mathcal{H}_{\mathcal{G}} \subset \mathcal{H}_{\text{aux}}$ of gauge invariant functions. It can be shown that this space is spanned by the so-called spin networks states [21,9].

If we try to follow the same procedure for the diffeomorphism constraint, we find that there are no non-trivial diffeomorphism invariant states in $\mathcal{H}_{\mathcal{G}}$. This problem is overcome by looking for distributional solutions to the constraints. Again the natural pull-back action of the diffeomorphism group on the connections induces a unitary representation of \mathcal{D} on \mathcal{H}_{aux} because of the diffeomorphism invariance of the inner product (2). One then considers a Gel'fand triple construction $F \subset \mathcal{H}_{\mathcal{G}} \subset F'$, where F is a dense subspace⁶ of $\mathcal{H}_{\mathcal{G}}$ and F' its topological dual and identifies the reduced Hilbert space \mathcal{H}_{kin} with a subspace of F' that is invariant under the dual action of the diffeomorphism constraint. \mathcal{H}_{kin} carries a natural dual action of the algebra $\mathfrak{B}_{\text{kin}}$,

⁶ F is usually chosen to be $\overline{\mathfrak{C}}$.

which is the subalgebra of $\mathfrak{B}_{\text{aux}}$ containing the elements that commute with the constraints.

2.4 Problems

As we stated in the introduction, we want to study states which represent asymptotically flat classical metrics, especially Minkowski space. We now argue that such states generically do not lie in the Hilbert space \mathcal{H}_{aux} constructed above, which is the reason that this representation is not adequate for non-compact Σ .

States which describe non-compact geometries should either be based on curves of infinite length or an infinite number of curves. This is because the area and volumes of regions of Σ which do not contain edges and vertices of graphs vanish. A particular example of an attempt to construct a state which approximates a chosen flat Euclidean 3-metric g_{ab} on Σ is the so-called weave given in [13]. This weave is based on an infinite collection of graphs $\Gamma_{r,\mu} = \bigcup_{i=1}^{\infty} D_i$, where D_i is the union of two randomly oriented circles γ_i^a and γ_i^b of radius r , which intersect in one point. To ensure isotropy these graphs are sprinkled randomly in Σ , with sprinkling density μ , where μ is defined with respect to g_{ab} . Given n double circles D_i we can construct the cylindrical function \mathcal{W}_n :

$$\mathcal{W}_n(A) = \prod_{i=1}^n \text{Tr}[\rho_1(H(\gamma_i^a, A)H(\gamma_i^b, A))] \quad (3)$$

where ρ_1 denotes the fundamental representation of $\text{SU}(2)$. The weave state \mathcal{W} should arise in the limit $n \rightarrow \infty$. This limit does not exist in the Hilbert space \mathcal{H}_{aux} , since the above sequence \mathcal{W}_n is not Cauchy in either the sup norm or the L^2 norm based on the inner product (2). This holds even if we impose physically reasonable (non-uniform) fall-off conditions on the connections, such as those for asymptotically flat gravity. The basic problem is that for any curve embedded in Σ we can always find a connection that will assign to this curve any holonomy we choose [7]⁷. In particular, this means that if we have a state $f(H(\gamma, A))$ then $\sup |f(H(\gamma, A))|$ will be independent of the location of γ . This is a generic result, and we conclude that a large class of physically interesting states based on infinite collections of graphs do not exist in \mathcal{H}_{aux} . Similar arguments can be used to show that states based on curves of infinite length do not lie in \mathcal{H}_{aux} either.

A very natural way of dealing with states based on a finite number of curves

⁷ We thank John Baez for this observation.

of infinite length was introduced in [1]. The key point is to consider a compactification of Σ and show that these states belong to the auxiliary Hilbert space constructed on the compactified manifold. Problems arise when trying to extend this approach to discuss cylindrical functions based on graphs with infinite number of edges and vertices as cluster points of vertices necessarily arise in the compactified manifold. This just illustrates the fact that the Hilbert space \mathcal{H}_{aux} , along with the representation of observables it carries, was constructed for compact spatial slices and is not adequate to describe the case of non-compact Σ . In the next section, we propose a different solution to the above problems by giving a procedure to construct new Hilbert spaces for quantum general relativity, which describe fluctuations around specified background states. In particular, these states can have non-compact support on the spatial manifold.

2.5 A new representation for $\mathfrak{B}_{\text{aux}}$

We take an approach analogous to that in algebraic field theory, where Hilbert spaces describing field theory at finite temperature arise as inequivalent representations of the algebra of observables via the GNS construction. In this approach, the algebra of observables is considered as primary, as opposed to the Hilbert space of states. This gives us the flexibility to consider different Hilbert spaces depending on which background state we are interested in. The vectors in this space then describe finite perturbations around this preferred state. In practice, we use the background state to define a positive linear form on our observable algebra $\mathfrak{B}_{\text{aux}}$ by interpreting the form as the expectation value of the observables in the preferred state. This then gives the starting point for the GNS construction which leads to the desired quantum theory. We have the following procedure:

- (1) The fact that we have a representation π_s of $\mathfrak{B}_{\text{aux}}$ on \mathcal{H}_{aux} as given in section 2.2 enables us to identify $\mathfrak{B}_{\text{aux}}$ with a subalgebra of the concrete $*$ -algebra⁸ of operators on \mathcal{H}_{aux}
- (2) Now we define a new positive linear form ω on $\mathfrak{B}_{\text{aux}}$, which is interpreted as the ‘vacuum’ expectation value of the elementary variables. Note that that in contrast to section 2.2 we are defining ω on all of $\mathfrak{B}_{\text{aux}}$ not just \mathfrak{C} .
- (3) Using the GNS construction we can now proceed to construct a representation of $\mathfrak{B}_{\text{aux}}$. The vectors in the carrier Hilbert space will be equivalence classes of elements of $\mathfrak{B}_{\text{aux}}$, which should be interpreted as excitations of

⁸ If one desires to work with a C^* -algebra one faces the problem that the ‘momentum’ operators are unbounded. To proceed one needs to consider algebras of bounded functions on $\mathfrak{B}_{\text{aux}}$ or consider families of spectral projectors of the unbounded operators. Physically, this does not lead to a loss of generality, c.f. [14].

the ‘vacuum’ state, obtained by acting on the ‘vacuum’ with the corresponding algebra element.

In the remainder of this paper we will demonstrate how we can construct ω explicitly. We do this by constructing a modified weave, which approximates flat space. As before the weave is not well defined as a state in \mathcal{H}_{aux} but it can be used to define ω , which will give the expectation values of the elements of $\mathfrak{B}_{\text{aux}}$ in this background state.

3 Approximating Classical Geometries

Let us start with a brief discussion of the problem of approximating classical metrics by quantum states. It is generally accepted that to obtain semi-classical behaviour from a quantum theory, one needs two things: i) a suitable coarse-graining, and, ii) coherent states. So far, however, coherent states have not been constructed in loop quantum gravity. To obtain these one needs to construct a state in which neither the 3-geometry nor its time-derivative are sharp. Rather, they should both have some minimum spreads as dictated by the uncertainty principle. We will come back to this point later.

An example of states which approximate classical 3-metrics are weave states (well-defined only if Σ is compact), such as the one defined in eq. (3) (see e.g. [13,6]). However, all weaves which have been constructed so far are eigenstates of the 3-geometry, so they are highly delocalized in their time derivative. Intuitively, this means that while a weave may approximate the 3-metric at one instant of time, evolving the state for even an infinitesimal time will completely destroy this approximation. In this section, we will construct a more satisfactory set of states that can also be used to approximate 3-metrics. In particular, these states can be used to define a positive linear form on $\mathfrak{B}_{\text{aux}}$ as is needed for the GNS construction even in the case that we want to approximate a non-compact geometry.

3.1 Approximating 3-metrics

Let us now take a closer look at the weaves. Their construction is made possible by the existence of operators on \mathcal{H}_{aux} which measure the area of a surface and the volume of a region [20,12,4,8]. This allows us to approximate classical metrics by requiring that the expectation values of areas and volumes of macroscopic surfaces and regions agree with the classical values.

For concreteness, in the rest of this section, we shall restrict ourselves to the

problem of approximating the flat Euclidean metric on \mathbb{R}^3 . Let w be a state which approximates the flat space at scales larger than a cut-off scale ℓ_c . The approximation problem can then be stated as follows: Given any object (of characteristic size larger than ℓ_c) with bulk R and surface S in \mathbb{R}^3 , we wish to make repeated measurements of the volume of the region $V[R]$ and the area of the surface $A[S]$ in the state w while placing the object at different points in space. If we wish to recover values corresponding to the flat metric g_{ab} on \mathbb{R}^3 at large scales, we require:

- (1) The average values of the area and volume for S and R obtained during the measurements should be given by the classical values:

$$\overline{\langle A_S \rangle} = A_g[S] \equiv \int_S \sqrt{\det {}^2g_{ab}} \quad (4)$$

$$\overline{\langle V_R \rangle} = V_g[R] \equiv \int_R \sqrt{\det g_{ab}} \quad (5)$$

where ${}^2g_{ab}$ denotes the induced 2-metric on S . Here a bar over the value indicates an average with respect to position in space whereas the angle brackets indicate the expectation value in the quantum state.

- (2) The standard deviation σ of the measurements should be small compared to the length scale ℓ_c accessible by current measurements:

$$\sigma_V \ll \ell_c^3 \quad \text{and} \quad \sigma_A \ll \ell_c^2, \quad (6)$$

where σ_V and σ_A denote the standard deviations in a series of measurements determining the volume and area of objects of the scale ℓ_c .

We shall call any state satisfying conditions (4), (5) and (6), a weave state. These conditions do not determine a state uniquely. Rather, one can construct infinitely many states which satisfy them. Below, we give an example of a weave state which is not an eigenstate of the three-geometry, but is peaked in both the connection and spin-networks bases. We refer to this state as a “quasi-coherent” weave in the following.

3.2 A “quasi-coherent” weave, \mathcal{Q}

Since a weave has to give areas and volumes to all surfaces in a non-compact manifold, the state must be based on an infinite graph. We take this graph to be $\Gamma_{r,\mu}$ as defined in section 2.4. The values of the parameters r and μ will be determined by the requirement that the state based on $\Gamma_{r,\mu}$ satisfy the weave conditions.

To define the state, we start with the cylindrical function q_i based on the

graph D_i in $\Gamma_{r,\mu}$:

$$q_i(A) = \eta \exp \left(\lambda \text{Tr} \left[\rho_1(H(\gamma_i^a, A)H(\gamma_i^b, A)) - \rho_1(e) \right] \right), \quad (7)$$

where λ is an arbitrary constant, e is the identity in $\text{SU}(2)$ and η is a normalisation factor. We also consider the products:

$$\mathcal{Q}_n(A) = \prod_{i=1}^n q_i(A).$$

The state that we are interested in, is the limit $\mathcal{Q} = \mathcal{Q}_\infty$, which is again not an element of the standard Hilbert space. Nevertheless, as we will see, this product can serve as a background for the construction of new Hilbert spaces, describing excitations of \mathcal{Q} .

Let us denote the connection that gives a trivial holonomy on all paths by A_0 . By construction the functions $\mathcal{Q}_n(A)$ take on their maximum values at A_0 . Conversely, knowledge of the holonomies on all paths in Σ allows us to determine a corresponding connection uniquely. Hence as $n \rightarrow \infty$ the function \mathcal{Q}_n becomes increasingly peaked around A_0 , the sharpness of the peak being determined by λ . This is one of the reasons for calling our weave a “quasi-coherent” state, the other being the exponential dependence on group elements which is characteristic of coherent states. We will explore these properties of the state further in future work. For the present, we are interested in showing that \mathcal{Q} is a good weave. In order to do so, we need to show that it satisfies the weave conditions (4), (5) and (6). We shall do this by demonstrating that standard deviations of area and volume measurements are roughly of the order of $\ell_c \ell_P$ and $\sqrt{(\ell_c \ell_P)^3}$, where $\ell_P = \sqrt{\hbar G_{\text{Newton}}/c^3}$ is the Planck length and hence much smaller than the bounds set by eq. (6). If only interested in how \mathcal{Q} can be used to construct new Hilbert spaces the reader may skip to the next section.

Let us start by expanding the state $q_i(A)$ in into an eigenbasis of the area operator and calculate the area expectation values and deviations. We do this by noting that the cylindrical function $f_p(A) = \text{Tr}[\rho_p(H(\gamma_i^a, A)H(\gamma_i^b, A))]$ based on the graph D_i is an eigenstate of the area operator⁹, where ρ_p is a representation of $\text{SU}(2)$ in ‘colour’ notation, i.e. $p = \dim(\rho) - 1$. The eigenvalues a_p of the area operator corresponding to some surface S , which intersects D_i *exactly once*, are given by $16\pi\ell_P^2\sqrt{\frac{p}{2}(\frac{p}{2}+1)}$. Thus, to evaluate the area expectation value $\langle a \rangle$ and the deviation Δ_a of this operator we start by expanding the function $q_i(A)$ in terms of the area eigenstates $f_p(A)$: $q_i = \sum_p s_p f_p$. We

⁹ This follows since f_p can be expanded in terms of spin-network functions that all assign ρ_p to each edge of D_i . Hence all spin-network functions in the expansion have the same area eigenvalues.

want to determine the coefficients of this expansion. We start by noting that q_i is defined by its series expansion (the i index labelling the graph will be suppressed in the following):

$$q = \eta e^{-2\lambda} \left(1 + \lambda f_1 + \frac{\lambda^2 f_1^2}{2!} + \frac{\lambda^3 f_1^3}{3!} + \dots \right).$$

Hence, we need to expand f_1^n in terms of f_p to determine the s_p 's. This can be done by using the decomposition rules for tensor products of representations of $SU(2)$:

$$\text{Tr}^n[\rho_1(g)] = \text{Tr}[\oplus_p c_p^n \rho_p(g)].$$

Hence, it follows that $f_1^n = \sum_p c_p^n f_p$. To determine the coefficients c_p^n , we use the fact that $\rho_p \otimes \rho_1 = \rho_{p-1} \oplus \rho_{p+1}$ and that:

$$\underbrace{\rho_1 \otimes \dots \otimes \rho_1}_n = (\oplus_p c_p^{n-1} \rho_p) \otimes \rho_1.$$

This gives us the following recursion relation:

$$c_p^n = c_{p-1}^{n-1} + c_{p+1}^{n-1}, \quad n, p \geq 0, \quad p \leq n.$$

Using the condition $c_0^0 = 1$, we can solve this recursion relation to get:

$$c_p^n = \frac{(p+1)n!}{\left(\frac{n-p}{2}\right)!\left(\frac{n+p}{2}+1\right)!} \quad \text{for } \frac{n-p}{2} \in \mathbb{N}, \quad (8)$$

and $c_p^n = 0$ otherwise. The expansion coefficients s_p (as a function of λ) are given by

$$s_p(\lambda) = \mathcal{N}(\lambda) \sum_{n=0}^{\infty} \frac{\lambda^n c_p^n}{n!}$$

where $\mathcal{N}(\lambda)$ is defined such that $\sum_{p=0}^{\infty} s_p^2 = 1$. Substituting eq. (8) into the right hand side of the above expression, we find:

$$\begin{aligned} s_p(\lambda) &= \mathcal{N}(\lambda) (p+1) \sum_{k=0}^{\infty} \frac{\lambda^{2k+p}}{k!(k+p+1)!} = \mathcal{N}(\lambda) \frac{(p+1)}{\lambda} I_{p+1}(2\lambda) \\ &= \mathcal{N}(\lambda) [I_p(2\lambda) - I_{p+2}(2\lambda)], \end{aligned}$$

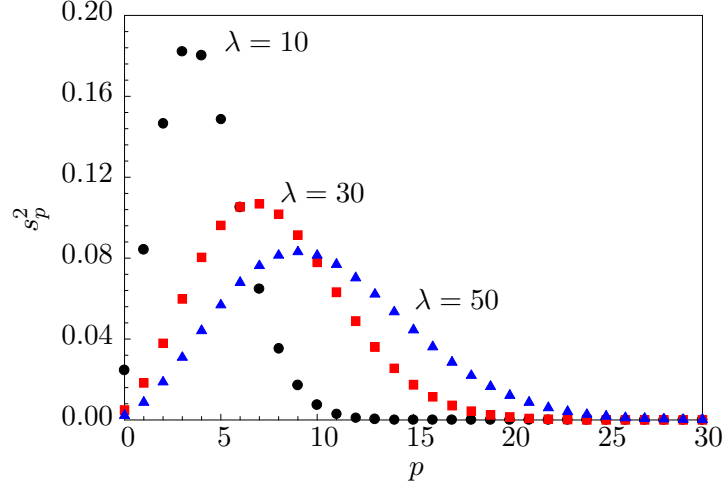


Fig. 1. Normalised coefficients s_p^2 for $\lambda = 10$, $\lambda = 30$ and $\lambda = 50$. The sharpness of the peak decreases with increasing λ .

where $I_p(x)$ is the modified Bessel function of order p . Using the properties of Bessel functions, we can then evaluate the normalisation constant to be

$$\mathcal{N}(\lambda) = [I_0(4\lambda) - I_2(4\lambda)]^{-1/2}.$$

Figure 1 shows the numerical values of the coefficients s_p^2 for a few values of λ .

The final form of the expansion of q_i in terms of the area eigenstates f_p is

$$q(A) = \sum_{p=0}^{\infty} \frac{I_p(2\lambda) - I_{p+2}(2\lambda)}{\sqrt{I_0(4\lambda) - I_2(4\lambda)}} f_p(A). \quad (9)$$

We needed to show that the states q_i are peaked in area and volume. Let us consider, in particular, $\lambda = 30$. In this case, we have:

$$\langle a \rangle = 4.33(16\pi\ell_P^2) \quad \text{and} \quad \Delta_a \equiv \sqrt{\langle a^2 \rangle - \langle a \rangle^2} = 1.87(16\pi\ell_P^2),$$

and hence both values are of order ℓ_P^2 . The explicit calculation for the volume expectation value and deviation, $\langle v \rangle$ and Δ_v , in the state q_i is somewhat more complicated since the states f_p have to be expanded in volume eigenstates. After doing this, we find:

$$\langle v \rangle = 3.61(16\pi\ell_P^2)^{3/2} \quad \text{and} \quad \Delta_v \equiv \sqrt{\langle v^2 \rangle - \langle v \rangle^2} = 2.21(16\pi\ell_P^2)^{3/2}.$$

Again, both values are of the order of a few ℓ_P^3 .

In general, the area expectation value for the area as a function of λ can be written as

$$\langle a \rangle = 16\pi\ell_P^2 \sum_{p=0}^{\infty} \frac{(I_p(2\lambda) - I_{p+2}(2\lambda))^2}{I_0(4\lambda) - I_2(4\lambda)} \sqrt{\frac{p}{2} \left(\frac{p}{2} + 1 \right)}.$$

We do not have a closed form for $\langle v \rangle$ since the evaluation of the volume expectation value involves the diagonalization of a matrix. However, since the largest eigenvalue of the volume operator in the state f_p increases with p as $p^{3/2}$, for a value of λ for which the expansion is dominated by a few small p 's, we can see that $\langle v \rangle$ has to be of the order of a few ℓ_P^3 . Given the fact that both $\langle a \rangle$ and $\langle v \rangle$ have value of the order of a few Planck units, it is reasonable to assume that the bound (6) is satisfied for area and volumes in the state \mathcal{Q} , which is a product of the q_i 's, as well. We now show that this is indeed the case.

The expectation value of the volume operator for a region \hat{V}_R depends upon the number N_R of double circles in the region R , namely:

$$\langle V_R \rangle = N_R \langle v \rangle.$$

Similarly, the expectation value for the area operator of a surface \hat{A}_S depends on the number of intersections N_S of the double circles with the surface:

$$\langle A_S \rangle = N_S \langle a \rangle.$$

Because the double circles D_i are sprinkled randomly in Σ the number of the graphs in any given region R of volume $V_g[R]$ is given by a Poisson distribution. In particular, the average value is given by $\bar{N}_R = V_g[R]\mu$. Hence average values of the above measurements are given by:

$$\overline{\langle V_R \rangle} = \bar{N}_R \langle v \rangle \tag{10}$$

and

$$\overline{\langle A_S \rangle} = \bar{N}_S \langle a \rangle, \tag{11}$$

where $\bar{N}_S = \frac{2}{3}V[S]\mu$ and $V[S] = 6rA_g[S]$ denotes the volume of a shell surrounding S with thickness $3r$ on either side. The factor $2/3$ is the average number of crossings between a double circle within this shell with S as determined in [13].

To determine the standard deviations σ_V and σ_A around $\overline{\langle V_R \rangle}$ and $\overline{\langle A_S \rangle}$ we use an approximation: we assume that the eigenvalues of \hat{V}_R and \hat{A}_S are given by

the product of two independent quantities, i.e. $N_R(V_R/\bar{N}_R)$ and $N_S(A_S/\bar{N}_S)$ respectively, where V_R and A_S are sums of any n elementary eigenvalues v_p and a_p . The deviations of the quantities N_R and N_S are given by $\sqrt{\bar{N}_R}$ and $\sqrt{\frac{2}{3}\bar{N}_S}$, since we are dealing with a Poisson distribution of graphs. Deviations of V_R/\bar{N}_R and A_S/\bar{N}_S on the other hand are $\Delta_v/\sqrt{\bar{N}_R}$ and $\Delta_a/\sqrt{\bar{N}_S}$. Our approximation implies that:

$$\begin{aligned}\sigma_A &\approx \sqrt{\bar{N}_S} \sqrt{\Delta_a^2 + \frac{2}{3}\langle a \rangle^2} \\ \sigma_V &\approx \sqrt{\bar{N}_R} \sqrt{\Delta_v^2 + \langle v \rangle^2}\end{aligned}$$

From eqs. (4), (5), (10), and (11), we find: $\bar{N}_S = A_g[S]/\langle a \rangle$ and $\bar{N}_R = V_g[R]/\langle v \rangle$. Because of the scale of $\langle a \rangle$, Δ_a and $\langle v \rangle$, Δ_v we conclude that eq. (6) is satisfied. Thus, we have shown that \mathcal{Q} is a weave state. For any particular value of λ , we can determine μ and r using eqs. (10), (11), (4) and (5):

$$\mu = \langle v \rangle^{-1} \quad \text{and} \quad r = (4\mu\langle a \rangle)^{-1}.$$

We will denote the infinite collection $\bigcup_{i=1}^{\infty} D_i$ of double circles D_i with r and μ determined by the above conditions by $\Gamma_{\mathcal{Q}}$.

4 New Hilbert Spaces

We now show how we can use \mathcal{Q} to define a new representation of the algebra $\mathfrak{B}_{\text{aux}}$ following the steps outlined in section 2.5. We begin by noting that we have a representation π_s of $\mathfrak{B}_{\text{aux}}$ on \mathcal{H}_{aux} , but as we have seen \mathcal{Q} does not belong to this Hilbert space. Nevertheless, we can define the action of an element of $\mathfrak{B}_{\text{aux}}$ on \mathcal{Q} . The crucial point is that the elementary quantum observables — the elements of \mathfrak{C} and the derivations on \mathfrak{C} — have support on a compact spatial region, which is a direct consequence of the smearing needed to make sense of the classical expressions. But if we restrict \mathcal{Q} to any compact region of Σ we obtain an element of \mathcal{H}_{aux} by restricting the underlying graph $\Gamma_{\mathcal{Q}}$ to that region. Hence given an arbitrary element $\mathbf{a} \in \mathfrak{B}_{\text{aux}}$ we proceed as follows:

- (1) Denote the closure of the support of \mathbf{a} by $R \subset \Sigma$.
- (2) Construct the graph $\Gamma_{\mathcal{Q}}|_R$ of $\Gamma_{\mathcal{Q}}$ restricted to R :

$$\Gamma_{\mathcal{Q}}|_R \equiv \bigcup_{D_i \cap R \neq \emptyset} D_i.$$

In other words, consider the union of all double circles which have a non-zero intersection with the support of \mathbf{a} . This graph is finite, since R is compact and the double circles D_i are sprinkled in Σ with finite density μ , we obtain the state $\mathcal{Q}|_R \in \mathcal{H}_{\text{aux}}$ which is given by restricting \mathcal{Q} to the graph $\Gamma_{\mathcal{Q}}|_R$:

$$\mathcal{Q}|_R \equiv \mathbb{I} \cdot \prod_{D_i \in \Gamma_{\mathcal{Q}}|_R} q_i,$$

where $\mathbb{I}(A) = 1$ for all A is the identity function. This state has unit norm in \mathcal{H}_{aux} since all the q_i 's are normalised.

- (3) Since $\mathcal{Q}|_R \in \mathcal{H}_{\text{aux}}$, the action of \mathbf{a} on $\mathcal{Q}|_R$ denoted by $\pi(\mathbf{a})_s \mathcal{Q}|_R$ is well-defined. It is understood that the region R will depend on \mathbf{a} .

This allows us to define $\omega_{\mathcal{Q}}(\mathbf{a})$:

$$\omega_{\mathcal{Q}}(\mathbf{a}) = \langle \mathcal{Q}|_R | \pi_s(\mathbf{a}) \mathcal{Q}|_R \rangle_s = \int \mathcal{Q}|_R^* \pi_s(\mathbf{a}) \mathcal{Q}|_R d\mu(A), \quad (12)$$

where the integral is defined as in eq. (1). This is well-defined since the integrand is an element of \mathcal{H}_{aux} . It follows from the fact that equations (2) defines a true inner product $\langle \cdot | \cdot \rangle_s$ on \mathcal{H}_{aux} that ω is indeed a positive (not necessarily strictly positive) linear form on $\mathfrak{B}_{\text{aux}}$.

Given this positive linear functional, we could proceed with steps 2 and 3 of the GNS construction outlined in section 2.2 to obtain a representation of the algebra $\mathfrak{B}_{\text{aux}}$. Instead, to get a more intuitive representation we make use of the theorem below to construct a unitarily equivalent representation $\pi_{\mathcal{Q}}$ of $\mathfrak{B}_{\text{aux}}$ on a Hilbert space $\mathcal{H}_{\mathcal{Q}}$ obtained by defining a new inner product on \mathfrak{C} .

Theorem 1 *Any representation π of a $*$ -algebra \mathfrak{A} with cyclic vector¹⁰ χ such that:*

$$\langle \chi | \pi(\mathbf{a}) \chi \rangle = \omega(\mathbf{a}),$$

for all $\mathbf{a} \in \mathfrak{A}$ is unitarily equivalent to the GNS representation π_{ω} , with cyclic vector χ_{ω} (corresponding to the unit element in \mathfrak{A}).

Proof The proof of this theorem is analogous to the one of proposition 4.5.3 in [17]. To proceed, we note that for each $\mathbf{a} \in \mathfrak{A}$,

$$\begin{aligned} \|\pi(\mathbf{a})\chi\|^2 &= \langle \pi(\mathbf{a})\chi | \pi(\mathbf{a})\chi \rangle = \langle \pi(\mathbf{a})\chi | \pi(\mathbf{a}^* \mathbf{a})\chi \rangle \\ &= \omega(\mathbf{a}^* \mathbf{a}) = \|\mathbf{a}\|_{\omega}^2 \end{aligned}$$

¹⁰ The vector $\chi \in \mathcal{H}$ is cyclic with respect to the representation π of \mathfrak{A} on \mathcal{H} if $\pi(\mathfrak{A})\chi$ is dense in \mathcal{H} .

where $\|\cdot\|_\omega^2$ is the norm in the Hilbert space \mathcal{H}_ω which carries the GNS representation. This means that there exists a norm-preserving, linear operator U_0 such that $U_0\pi(\mathbf{a})\chi = \pi_\omega\chi_\omega$. Since χ is a cyclic vector, U extends by continuity to an isomorphism from the Hilbert space \mathcal{H} to \mathcal{H}_ω and $U\chi_\omega = \chi$.

For any $\mathbf{a}, \mathbf{b} \in \mathfrak{A}$,

$$\begin{aligned} U\pi_\omega(\mathbf{a})\pi_\omega(\mathbf{b})\chi_\omega &= U\pi_\omega(\mathbf{ab})\chi_\omega = \pi(\mathbf{ab})\chi \\ &= \pi(\mathbf{a})\pi(\mathbf{b})\chi = \pi(\mathbf{a})U\pi_\omega(\mathbf{b})\chi_\omega. \end{aligned}$$

Since χ_ω is cyclic in \mathcal{H}_ω , we have $U\pi_\omega(\mathbf{a}) = \pi(\mathbf{a})U$ which in turn implies that $\pi(\mathbf{a}) = U\pi_\omega(\mathbf{a})U^*$. \square

Using the above theorem, we proceed as follows:

- (1) On \mathfrak{C} , the space of cylindrical functions, introduce the following strictly positive inner product:

$$\langle \psi_{f_1, \Gamma_1} | \psi_{f_2, \Gamma_2} \rangle_{\mathcal{Q}} = \int \mathcal{Q}|_R^* \mathcal{Q}|_R \psi_1^* \psi_2 \, d\mu(A), \quad (13)$$

where R here is the union of the graphs Γ_1 and Γ_2 . Completion of \mathfrak{C} with respect to this positive definite inner product gives us the Hilbert space $\mathcal{H}_{\mathcal{Q}}$.

- (2) We construct a representation $\pi_{\mathcal{Q}}$ of $\mathfrak{B}_{\text{aux}}$ on \mathfrak{C} , which is dense in $\mathcal{H}_{\mathcal{Q}}$ by:

$$\pi_{\mathcal{Q}}(\mathbf{a})\psi = \mathcal{Q}|_R^{-1} \pi_s(\mathbf{a})(\mathcal{Q}|_R \psi), \quad (14)$$

where $\mathbf{a} \in \mathfrak{B}_{\text{aux}}$, $\psi \in \mathfrak{C}$ and $\mathcal{Q}|_R^{-1}$ denotes the inverse function of $\mathcal{Q}|_R$ i.e., $\mathcal{Q}|_R^{-1} \mathcal{Q}|_R = \mathbb{I}$. At this point we note an additional requirement for the background state: $\mathcal{Q}(A) \neq 0$ for all A . This invertability property is motivated physically since our background state is meant to represent an infinite ‘condensate of gravitons’. We should be able to annihilate as well as create these gravitons, which motivates invertability. The definition (7) of $q_i(A)$ was chosen to satisfy this property.

It is now straightforward to see that $\pi_{\mathcal{Q}}$ is unitarily equivalent to the GNS representation defined via $\omega_{\mathcal{Q}}$ given in equation (12). We note that \mathbb{I} is a cyclic vector in $\mathcal{H}_{\mathcal{Q}}$ with respect to the representation $\pi_{\mathcal{Q}}$, since $\mathcal{H}_{\mathcal{Q}}$ is the closure of \mathfrak{C} . In addition we have:

$$\langle \mathbb{I} | \pi_{\mathcal{Q}}(\mathbf{a}) \mathbb{I} \rangle_{\mathcal{Q}} = \int \mathcal{Q}|_R^* \mathcal{Q}|_R \mathcal{Q}|_R^{-1} \pi(\mathbf{a})_s \mathcal{Q}|_R \, d\mu(A) = \omega_{\mathcal{Q}}(\mathbf{a}),$$

for all $\mathbf{a} \in \mathfrak{B}_{\text{aux}}$.

Hence we have constructed a Hilbert space and representation of observables on it that describes fluctuations restricted to essentially compact regions around some fixed infinite background state. Intuitively, the above representation has a clear interpretation. It acts on states by composing them with the background. We can regard the algebra of cylindrical functions \mathfrak{C} as creating and annihilating excitations on the background state. This new representation $\pi_{\mathcal{Q}}$ on $\mathcal{H}_{\mathcal{Q}}$ is related to the standard representation π_s on \mathcal{H}_{aux} given in section 2.2 by equation (14), i.e.: $\mathcal{Q}|_R \pi_{\mathcal{Q}}(\mathbf{a}) = \pi_s(\mathbf{a}) \mathcal{Q}|_R$. Since $\mathcal{Q}|_R$ depends on the algebra element \mathbf{a} , which is a direct consequence of \mathcal{Q} not being an element of \mathcal{H}_{aux} , this does not give us a unitary map between Hilbert spaces, instead we have many different maps. It follows that the representations $\pi_{\mathcal{Q}}$ and π_s are unitarily *inequivalent*¹¹. The construction we have presented is very general and can be applied to a large class of background states provided they satisfy the necessary invertability condition.

4.1 Constraints and Asymptotic Symmetries

We can proceed to reduce the Hilbert space obtained in the previous section by imposing the Gauss and diffeomorphism constraints. In general, when considering GNS representations π of an algebra \mathfrak{A} we can implement actions of symmetry groups G using the following theorem (eq. III.3.14 in [14]):

Theorem 2 *Given an action of G on \mathfrak{A} : $\mathbf{a} \rightarrow g\mathbf{a}$ such that $\omega(g\mathbf{a}) = \omega(\mathbf{a})$ for all $\mathbf{a} \in \mathfrak{A}$ and $g \in G$ then we can define a unitary representation U of G on \mathcal{H}_{ω} by:*

$$U(g)\pi(\mathbf{a})\chi_{\omega} = \pi(g\mathbf{a})\chi_{\omega}$$

where χ_{ω} is a cyclic vector in \mathcal{H}_{ω} .

In practice, when considering the representation $\pi_{\mathcal{Q}}$ we can proceed as in section 2.3 to reduce the Hilbert space $\mathcal{H}_{\mathcal{Q}}$. As before, there is no problem in implementing the Gauss constraint. Since the state $\mathcal{Q}|_R$ is invariant under gauge transformations, the inner product defined in eq. (13) is also gauge invariant and we have a unitary action of the gauge group on the state space. Again we implement the Gauss constraint by restricting to the subspace of $\mathcal{H}_{\mathcal{Q}}$ consisting of gauge invariant states, which is spanned by spin-networks.

¹¹ This can be also be seen by considering the following example. The state $\mathbb{I} \in \mathcal{H}_{\text{aux}}$ has the property that it is annihilated by *all* “momentum” operators. If there were a unitary map $U : \mathcal{H}_{\text{aux}} \rightarrow \mathcal{H}_{\mathcal{Q}}$ then $U\mathbb{I} \equiv \Psi \in \mathcal{H}_{\mathcal{Q}}$ should have the same property, i.e.: $\mathcal{Q}|_{\bar{S}}^{-1} \pi_s(E_{t,S})(\mathcal{Q}|_{\bar{S}} \Psi) = 0$ for all derivations $E_{t,S}$. But since Ψ has essentially compact support in Σ and since we can chose S so that $\pi_s(E_{t,S})\mathcal{Q}|_{\bar{S}} \neq 0$ outside any compact region, this cannot be satisfied for all regions S .

When considering diffeomorphisms, we notice that $\mathcal{Q}|_R$ is invariant only under diffeomorphisms that leave the graph $\Gamma_{\mathcal{Q}}$, on which \mathcal{Q} is based, invariant. Let us denote this subgroup of diffeomorphisms by Diff_{Γ} . Invariance of the inner product under Diff_{Γ} gives us a unitary representation of this group and we can use the Gel'fand triple construction detailed earlier to obtain a space of kinematical states $\mathcal{H}_{\mathcal{Q}}^{\text{kin}}$ that is invariant under Diff_{Γ} . This space then naturally carries a dual representation of the subalgebra $\mathfrak{B}_{\text{kin}} \subset \mathfrak{B}_{\text{aux}}$ of operators that commute with constraints.

To discuss the significance of the breaking of diffeomorphism invariance to the group Diff_{Γ} , we note that given two different background states \mathcal{Q} and \mathcal{Q}' which are defined with respect to diffeomorphic graphs: $\Gamma_{\mathcal{Q}'} = \phi \circ \Gamma_{\mathcal{Q}}$, where ϕ is any diffeomorphism, we obtain unitarily equivalent representations of $\mathfrak{B}_{\text{aux}}$:

$$U[\pi_{\mathcal{Q}}(\mathbf{a})\mathbb{I}] = \pi_{\mathcal{Q}'}(\phi\mathbf{a})\mathbb{I},$$

where unitarity of U follows from the diffeomorphism invariance of the inner product given by (2). Hence, the choice of a particular member of the diffeomorphism class of \mathcal{Q} is simply a partial gauge fixing. A physically equivalent way of getting the same results would be to average over the group Diff_{Γ} .

To conclude, we note that in the context of asymptotically flat general relativity, which is the prime case of interest involving non-compact spatial manifolds, the invariance group of the theory is restricted to the connected component of the asymptotically trivial diffeomorphisms. In the neighbourhood of infinity we would like to have a unitary action of the Poincaré group on our state space. Since physically relevant operators are typically evaluated at infinity, this invariance is what is of prime interest.

We shall discuss the construction of the action of the full Poincaré group in future work. Here, we show how a unitary action of the Euclidean group E acting on Σ can be incorporated in our scheme. From theorem 2, it follows that to do this we need a linear form on the algebra of observables that is invariant under the action of E . Such a form can be obtained by using the fact that the Euclidean group is locally compact to group average the form ω given in eq. (12). Given an increasing sequence of compact subsets $S_k \subset E$, $S_k \subset S_{k+1}$, $\cup S_k = E$ we define:

$$\omega_k(\mathbf{a}) = \mu(S_k)^{-1} \int_{S_k} \omega(g\mathbf{a}) d\mu(g),$$

where $d\mu(g)$ is the invariant measure on E and $g \in E$. It can be shown [14] that the sequence ω_k converges to a positive linear form ω_E , which is invariant under E . Using this form we obtain a representation of $\mathfrak{B}_{\text{aux}}$ on a state

space \mathcal{H}_E carrying a unitary representation of E . Equivalently, we could have used this procedure to average the background state \mathcal{Q} , to obtain the desired representation.

5 Conclusions and future directions

In this work, we have presented two main results. The first of these is a general procedure for construction of new Hilbert spaces for loop quantum gravity for non-compact spaces. We used an analogy with thermal field theory to construct a non-standard representation of the classical algebra of observables. A key ingredient for this construction was the use of a background state which is analogous to the thermal ground state.

We also presented a possible candidate which can be used as a background state. This is a weave state in that it approximates the classical flat Euclidean metric on \mathbb{R}^3 . The advantage of this state over previous weave constructions is that it is not an eigenfunction of the three-geometry. Rather, it is peaked both in the connection and the spin-network pictures. This was our second main result.

We would like to conclude with a discussion of some open issues and future directions.

- (1) The properties of the state \mathcal{Q} deserve to be better studied. In particular, we would like to investigate any possible relation between \mathcal{Q} and coherent states which may be defined on the group $SU(2)$.
- (2) We would also like to study the low-energy sector of the Hilbert space constructed with \mathcal{Q} as a background state and look at its relation to the standard Fock space of gravitons. In this context, it would also be interesting to understand the connection of our work to [15,16].
- (3) We are in the process of computing the spectra of the area and volume operators in the new Hilbert space defined by \mathcal{Q} to verify the intuitive picture of areas and volumes fluctuating around flat space values.
- (4) A quantum positivity of energy theorem was proved in [22]. However, as we have shown, the Hilbert space on which that result was proved is not applicable to the study of non-compact spatial geometries. We believe that our construction provides the proper arena for questions of this nature and are currently investigating the properties of suitably defined ADM energy and momentum operators on our Hilbert space.

This work is a step in the direction of making contact between the non-perturbative quantisation of gravity and the picture of graviton physics which arises from standard perturbative quantum field theory. A lot more work needs

to be done before the relation between the two is completely clarified.

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References

- [1] M. Arnsdorf and R.S. Garcia. Existence of spinorial states in pure loop quantum gravity. *Class. Quant. Grav.*, 16:3405–3418, 1999, e-print: gr-qc/9812006.
- [2] A. Ashtekar. New variables for classical and quantum gravity. *Phys. Rev. Lett.*, 57:2244, 1986.
- [3] A. Ashtekar, A. Corichi, and J.A. Zapata. Quantum theory of geometry III: Non-commutativity of riemannian structures. 1998, e-print: gr-qc/9806041.
- [4] A. Ashtekar and J. Lewandowski. Quantum theory of geometry I: Area operators. *Class. Quant. Grav.*, 14:A43, 1997, e-print: gr-qc/9602046.
- [5] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão, and T. Thiemann. Quantization of diffeomorphism invariant theories of connections with local degrees of freedom. *J. Math. Phys.*, 36:6456, April 1995, e-print: gr-qc/9504018.
- [6] A. Ashtekar, C. Rovelli, and L. Smolin. Weaving a classical geometry with quantum threads. *Phys. Rev. Lett.*, 69:237, 1992.
- [7] Abhay Ashtekar and Jerzy Lewandowski. Representation theory of analytic holonomy c^* algebras. In J. Baez, editor, *Knots and Quantum Gravity*, Oxford, 1994. Oxford University Press, e-print: gr-qc/9311010.
- [8] Abhay Ashtekar and Jerzy Lewandowski. Quantum theory of geometry II: Volume operators. *Adv. Theor. Math. Phys.*, 1:388, 1998, e-print: gr-qc/9711031.
- [9] J.C. Baez. Spin networks in nonperturbative quantum gravity. In Louis Kauffman, editor, *The Interface of Knots and Physics*, page 167, Providence, Rhode Island, 1996. American Mathematical Society, e-print: gr-qc/9504036.
- [10] J.C. Baez and S. Sawin. Functional integration on spaces of connections. *Jour. Func. Anal.*, 150:1, 1997, e-print: q-alg/9507023.

- [11] F. Barbero. Real ashtekar variables for lorentzian signature space-times. *Phys. Rev.*, D 51:5507, 1995, e-print: gr-qc/9410014.
- [12] R. DePietri and C. Rovelli. Geometry eigenvalues and scalar product from recoupling theory in loop quantum gravity. *Phys. Rev.*, D54:2664, 1996, e-print: gr-qc/9602023.
- [13] N. Grot and C. Rovelli. Weave states in loop quantum gravity. *Gen. Rel. Grav.*, 29:1039, 1997.
- [14] R. Haag. *Local Quantum Physics*. Springer-Verlag, 1993.
- [15] J. Iwasaki and C. Rovelli. Gravitons as embroidery on the weave. *Int. J. Mod. Phys.*, D1:533, 1993.
- [16] J. Iwasaki and C. Rovelli. Gravitons from loops: non-perturbative loop space quantum gravity contains the graviton-physics approximation. *Class. Quantum Gravity*, 11:1653, 1994.
- [17] R.V. Kadison and J.R. Ringrose. *Fundamentals of the Theory of Operator Algebras*, volume 1. Academic Press, 1983.
- [18] N.P. Landsman and Ch.G. Weert. Real and imaginary-time field theory at finite temperature and density. *Phys. Rep.*, 145(3 & 4):141, 1987.
- [19] C. Rovelli and L. Smolin. Loop representation of quantum general relativity. *Nucl. Phys.*, B331:80, 1990.
- [20] C. Rovelli and L. Smolin. Discreteness of area and volume in quantum gravity. *Nucl. Phys.*, B442:593, 1995.
- [21] C. Rovelli and L. Smolin. Spin networks and quantum gravity. *Phys. Rev.*, D52:5743, 1995.
- [22] T. Thiemann. QSD VI: Quantum poincaré algebra and a quantum positivity of energy theorem for canonical quantum gravity. *Class. Quant. Grav.*, 15:1463, 1998, e-print: gr-qc/9705020.